Mathematics for Computer Science

TD4

October 1st, 2025

Question. *Write the proof that $m_n = \cdots$*

Before proving the result, we will state two useful lemmas.

Lemma 1. *Let $(m,n) \in \mathbb{N} \times \mathbb{Z}$, such that $m+n$ is even. We define*

$$\mathscr{P}_{m,n} \overset{\text{def}}{=} \{\textit{paths from } (0,0) \textit{ to } (m,n) \textit{ with steps } \nearrow \textit{ and } \searrow\}$$

and $p_{m,n} \overset{\text{def}}{=} |\mathscr{P}_{m,n}|$. Then $p_{m,n} = \binom{m}{\frac{n+m}{2}}$.

*Proof of **lemma 1**.* Let $m,n \in \mathbb{Z}$, $m \geq 0$ and $m+n$ even. For all path $\mathrm{p} \in \mathscr{P}_{m,n}$, each step of p makes us move 1 unit to the right. Since p reaches m on the x-axis, p is constituted of m steps in total. Moreover, let us define $\alpha \overset{\text{def}}{=} |\{\nearrow \text{ steps in p}\}|$ and $\beta \overset{\text{def}}{=} |\{\searrow \text{ steps in p}\}|$. We observe that $\alpha + \beta$ is equal to the total number of steps in p, which is m, and $\alpha - \beta$ gives the altitude of the arrival point of the path, which is n. Therefore, α, β solves a linear system $(\mathbb{S})$ which is equivalent to

$$(\mathbb{S}): \left\{ \begin{array}{rcl} \alpha+\beta & = & m \\ \alpha-\beta & = & n \end{array} \right. \qquad \Longleftrightarrow \qquad (\mathbb{S}): \left\{ \begin{array}{rcl} \alpha & = & (m+n)/2 \\ \beta & = & (m-n)/2. \end{array} \right.$$

This tells us that any path in $\mathscr{P}_{m,n}$ has exactly $(m+n)/2$ up-right steps $\nearrow$ and $(m-n)/2$ down-right steps $\searrow$. Therefore, there is as many paths in $\mathscr{P}_{m,n}$ as ways of choosing the order of the $(m+n)/2$ $\nearrow$ steps among the m steps, there is thus $\binom{m}{\frac{n+m}{2}}$ possible ways of building a path from $(0,0)$ to (m,n). □

As an immediate corrolary of **lemma 1**, we can now count how many paths links $(0,0)$ to the point $(2n,0)$: this number is

$$p_{2n,0} = |\mathscr{P}_{2n,0}| = \binom{2n}{\frac{2n+0}{2}} = \binom{2n}{n}.$$

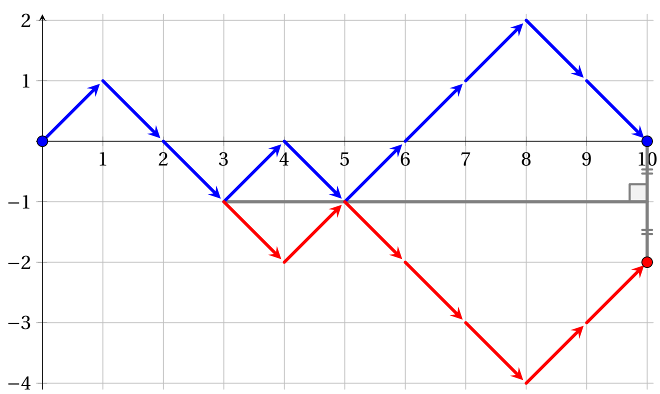
Lemma 2. *The sets $\mathscr{P}_{2n,0} \setminus \mathscr{M}_n$ and $\mathscr{P}_{2n,-2}$ are in bijection.*

*Proof of **lemma 2**.* Let $\mathrm{p} \in \mathscr{P}_{2n,0} \setminus \mathscr{M}_n$: it is a path from $(0,0)$ to $(2n,0)$ that is not a mountain, and in particular, p reaches a point at "altitude" -1 at least once. Let $a \in \mathbb{N}$ be the coordinate on the x-axis of the ***first point* at altitude -1**. The *mirror* operation consists of constructing the mirrored path, denoted by $\overline{\mathrm{p}}$, which is defined as follows: the path is constituted of the first a steps of p, and the last $2n-a$ steps are "reflected" by replacing each $\searrow$ step with an $\nearrow$ step and vice versa.

Geometrically, the mirrored path is formed by taking the first steps until it reaches an altitude of -1, and then applying the $x=-1$ symmetry axis.

For example, in the figure shown on the side, the mirrored path is consitituted of the first 3 blue steps, and the last 7 red steps.

The effect of this "symmetry point of view" is that, the destination ● of the mirrored path $\overline{\mathrm{p}}$ is the symmetric of the destination ● of the original path p around the line $x=-1$. Since, by definition, p ends in $(2n,0)$, the mirrored path $\overline{\mathrm{p}}$ must end in $(2n,-2)$.



Thus, the mirror operation $\mathbf{m} : \mathrm{p} \mapsto \overline{\mathrm{p}}$ maps any non-moutain path to a path that ends in $(2n, -2)$. We have to show that $\mathbf{m}$ is a bijection. Briefly, if p_1 and p_2 are two non-mountain paths that gets reflected to the same paths in $\mathscr{P}_{2n,-2}$, then $\mathrm{p}_1 = \mathrm{p}_2$. In other terms, the application $\mathbf{m}$ is *injective*.

Conversly, each path $\mathrm{q} \in \mathscr{P}_{2n,-2}$ must cross the altitude -1 at least once. Taking $b \in \mathbb{N}$ the coordinate on the x-axis of the first point of q reaching altitude -1, the same mirror operation gives a reflection $\overline{\mathrm{q}}$ that is a non-mountain path which has $(2n, 0)$ as destination. It means that each path $\mathrm{q} \in \mathscr{P}_{2n,-2}$ comes from a non-mountain path in $\mathscr{P}_{2n,0} \setminus \mathscr{M}_n$. In other terms, the application $\mathbf{m}$ is *surjective*.

Finally, the mirror operation $\mathbf{m} : \mathrm{p} \mapsto \overline{\mathrm{p}}$ is a bijection $\mathscr{P}_{2n,0} \setminus \mathscr{M}_n \xrightarrow{\simeq} \mathscr{P}_{2n,-2}$. □

We are now ready to compute m_n the number of moutains.

Proof. Since $\mathscr{P}_{2n,0} \setminus \mathscr{M}_n$ and $\mathscr{P}_{2n,-2}$ are finite sets, that are in bijection (by **lemma 2**), they share the same cardinal. We thus have:

$$\left|\mathscr{P}_{2n,0} \setminus \mathscr{M}_n\right| = \left|\mathscr{P}_{2n,-2}\right|$$
$$\underbrace{\left|\mathscr{P}_{2n,0}\right|}_{\stackrel{\text{def}}{=} p_{2n,0}} - \underbrace{\left|\mathscr{M}_n\right|}_{\stackrel{\text{def}}{=} m_n} = \underbrace{\left|\mathscr{P}_{2n,-2}\right|}_{\stackrel{\text{def}}{=} p_{2n,-2}}$$

and therefore, $m_n = p_{2n,0} - p_{2n,-2}$. Thanks to **lemma 1**, both this quantities can be easely computed, and we have $p_{2n,0} = \binom{2n}{n}$ and $p_{2n,-2} = \binom{2n}{\frac{2n-2}{2}} = \binom{2n}{n-1}$. Finally, we have:

$$m_n = \binom{2n}{n} - \binom{2n}{n-1}.$$

□

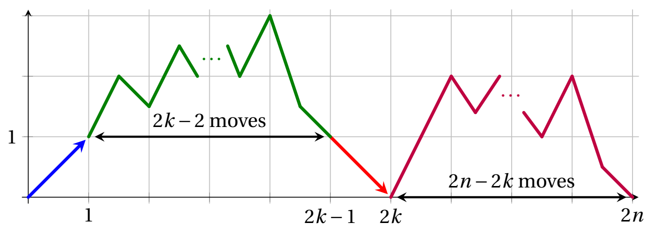
Remark. *To go further on this subject, we can give the arguments to prove the recursive formula:*

$$m_0 = 1 \qquad \textit{and} \qquad \forall n \in \mathbb{N}^*, \qquad m_n = m_0 m_{n-1} + m_1 m_{n-2} + \cdots + m_{n-2} m_1 + m_{n-1} m_0.$$

Let us consider a mountain $\mathrm{m} \in \mathscr{M}_n$. We know that the path must begin with a $\nearrow$ move; otherwise, it would immediately reach a negative altitude. Moreover, within the sequence of moves, there must exist at least one $\searrow$ move that decreases the altitude from 1 to 0. Let us focus on $\searrow$, the ***first move* that takes the altitude from 1 to 0**.

Since each move changes the altitude by either $+1$ or -1, the parity of the altitude coincides with the parity of the move number. The first move — an odd-numbered one — leads to altitude 1, which is odd. The second move — of even index — leads either to altitude 2 or 0, both even; and so forth. Therefore, if $\searrow$ is the ℓ^{th} move, then, since it reaches an even altitude (0), we must have that ℓ is even. Consequently, there exists some $k \in [\![n]\!]$ such that $\searrow$ is the $(2k)^{\text{th}}$ move.

Without loss of generality, our mountain can be represented as:



In the sequence of green moves, the altitude always remains above 1; otherwise, this would contradict the assumption that $\searrow$ is the first move leading to altitude 0. Consequently, the green sequence defines a mountain of length $2k-2$ moves, *i.e.* $(\text{green moves}) \in \mathscr{M}_{k-1}$, and this mountain lies at altitude $+1$. The same reasoning applies to the sequence of purple moves: it defines a sequence of movements of length $2n - 2k$ that goes from altitude 0 back to altitude 0 without crossing the x-axis; therefore, $(\text{purple moves}) \in \mathscr{M}_{n-k}$.

Now, let us construct a mountain of total length $2n$ whose first $\searrow$ occurs at position $2k$. To do so, we must select a mountain $\mathscr{M}_{k-1}$ to be inserted between $\nearrow$ and $\searrow$, and another mountain $\mathscr{M}_{n-k}$ to follow $\searrow$. Since there are m_{k-1} and m_{n-k} possible choices for these two sequences of moves, respectively, there are exactly $m_{k-1} \times m_{n-k}$ distinct ways to construct a mountain whose first $\searrow$ appears at position $2k$.

We now have a partition of the set of mountains as

$$\mathscr{M}_n = \bigsqcup_{k=1}^{n} \{\mathrm{m} \in \mathscr{M}_n \text{ such that } \searrow \text{ is in position } 2k\}$$

where the symbol "$\sqcup$" denotes a disjoint union. This immediately yields

$$\begin{aligned} m_n = |\mathscr{M}_n| &= \sum_{k=1}^{n} \left|\left\{ \mathrm{m} \in \mathscr{M}_n \text{ such that } \searrow \text{ is in position } 2k \right\}\right| \\ &= \sum_{k=1}^{n} m_{k-1} m_{n-k}. \end{aligned}$$